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Exploding soliton and front solutions of the complex cubic–quintic Ginzburg–Landau equation

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Abstract

We present a study of exploding soliton and front solutions of the complex cubic–quintic Ginzburg–Landau (CGLE) equation. We show that exploding fronts occur in a region of the parameter space close to that where exploding solitons exist. Explosions occur when eigenvalues in the linear stability analysis for the ground-state stationary solitons have positive real parts. We also study transition from exploding fronts to exploding solitons and observed extremely asymmetric soliton explosions.

Keywords: Ginzburg–Landau equation; Dissipative soliton; Exploding soliton

1. Introduction

Localized solutions in dissipative systems reveal some unusual properties that are unknown for such solutions in conservative systems. One well-studied model for dissipative solitons is the cubic–quintic complex Ginzburg–Landau equation (CGLE). This equation contains the basic terms describing the most important physical phenomena occurring in passively mode-locked lasers [1]. The CGLE also serves as a generic equation describing systems near sub-critical bifurcations [2,3]. It relates to a wide

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range of dissipative phenomena in physics, such as binary fluid convection [4], electro-convection in nematic liquid crystals [5], patterns near electrodes in gas discharges [6] and oscillatory chemical reactions [7].

One of the most recent discoveries related to this model is that of the so-called “exploding solitons”. These were found in numerical simulations [8,9] and their existence has been experimentally confirmed in a passively mode-locked solid state laser [10]. These solitons possess the interesting property of exploding at a certain point, breaking down into multiple pieces, and subsequently recovering their original shape. As noted above, exploding solitons can, in principle, be observed in a variety of applications.

Already in the first work on exploding (erupting) solitons [8], it was found that these localized objects exist over a wide range of the system parameters (see Fig. 5 of [8]). A quick comparison with the results of [11] shows that this range of parameters is, if not larger, then at least comparable with the range where stable stationary solitons exist. On the one hand, this remarkable property should make their observation a relatively easy task. On the other hand, we need to find the reasons why it happens.

Some explanations for the existence of exploding solitons and their unusual dynamics have been presented in Ref. [12]. Namely, the stability of the soliton in the laminar stage of evolution, i.e. in the state when the soliton enters its stationary regime of propagation, has been investigated in detail. This study revealed the structure of eigenvalues and eigenfunctions of the stationary soliton that causes the soliton to explode and then return to the same state afterwards.

In this work, we extend these studies and cover the case of front solutions. We have found, for the first time, that front solutions can also reveal chaotic behavior similar to the exploding solitons. Their shape can intermittently be covered with chaotic structures that tend to decrease after certain propagation distance. The range of parameters where fronts explode is very close to the range where solitons have explosive instability. Moreover, there is a continuous transition from exploding solitons to exploding fronts. This means that the nature of explosions is similar in each case. We give here explanations for the soliton explosions and assume that the same study can be done in the case of the exploding fronts. A detailed study for the fronts will be given elsewhere.

Another interesting observation that we made numerically is the existence of extremely asymmetric explosions. These happen mostly at one side of the soliton in spite of the fact that the equation, the soliton and the initial conditions are symmetric relative to the *t* variable. The side of the soliton where the explosion occurs alternate, so that explosions occur at the left and right sides of the soliton successively. We give explanations for this unusual phenomenon.

2. Master equation

The cubic–quintic complex Ginzburg–Landau equation can be written:

$$\mathrm{i}\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi + \nu|\psi|^4\psi = \mathrm{i}\delta\psi + \mathrm{i}\epsilon|\psi|^2\psi + \mathrm{i}\beta\psi_{tt} + \mathrm{i}\mu|\psi|^4\psi. \quad (1)$$

When used to describe passively mode-locked lasers, z is the cavity round-trip number, t is the retarded time, ψ is the normalized envelope of the field, D is the group velocity dispersion coefficient, with $D = \pm 1$, depending on whether the group velocity dispersion (GVD) is anomalous or normal, respectively, δ is the linear gain-loss coefficient, $\mathrm{i}\beta\psi_{tt}$ accounts for spectral filtering ($\beta > 0$), $\epsilon|\psi|^2\psi$ represents the nonlinear gain (which arises, e.g., from saturable absorption), the term with μ represents, if negative, the saturation

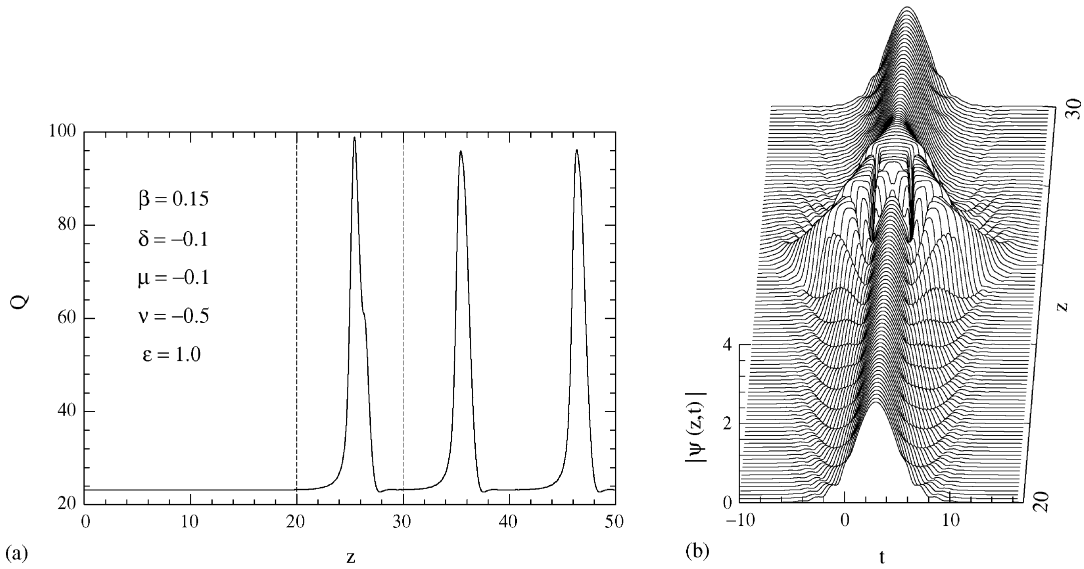

Fig. 1. (a) Energy of the exploding soliton at $\epsilon = 1.0$ vs. the propagation distance z. (b) Evolution of the soliton profile for one of the explosions: the one bounded by the vertical lines in (a).

of the nonlinear gain, while the one with ν corresponds, also if negative, to the saturation of the nonlinear refractive index.

Eq. (1) has a variety of localized solutions. These are stationary solitons, sources, sinks, moving solitons and fronts with fixed velocity [13,14]. A multiplicity of solutions can exist simultaneously. For example, solitons can exist in several forms and many of them can be stable for a certain range of values of the equation parameters [15]. In addition to localized solutions with fixed shape there are pulsating solitons [9], whose profile changes periodically, with the propagation distance z. Another interesting discovery is the "exploding soliton" [8]. This localized solution belongs to the class of chaotic solutions. This solution has intervals of almost stationary propagation, but, over and over, the instability develops, producing explosions, to recover its stationary shape subsequently. An example of exploding soliton is shown in Fig. 1.

The essential features of explosions, observed both theoretically [8,9] and experimentally [10], are: (1) Explosions occur intermittently. In the continuous model, they happen more or less regularly, but the period changes dramatically with a change of parameters. (2) The explosions have similar features, but are not identical. (3) Explosions happen spontaneously, but additional perturbations can trigger them. (4) One of the basic features of this solution is that the recurrence is back to the stationary soliton solution.

For the set of parameters shown in Fig. 1, explosions exist in a wide range of values of ϵ. Namely, we can change ϵ from 0.44 up to 1.68 and the soliton solution will have the properties listed above. At the values of ϵ higher than 1.68, the exploding soliton is transformed first into a pulsating solution and then into a stable stationary soliton. On the other hand, when ϵ is below 0.44, it is transformed into a pair of moving fronts which also have the property of exploding. Front explosions occur in a wide range of ϵ which extends from 0.3 to 0.4. In the present paper, we report the first observation of these "exploding

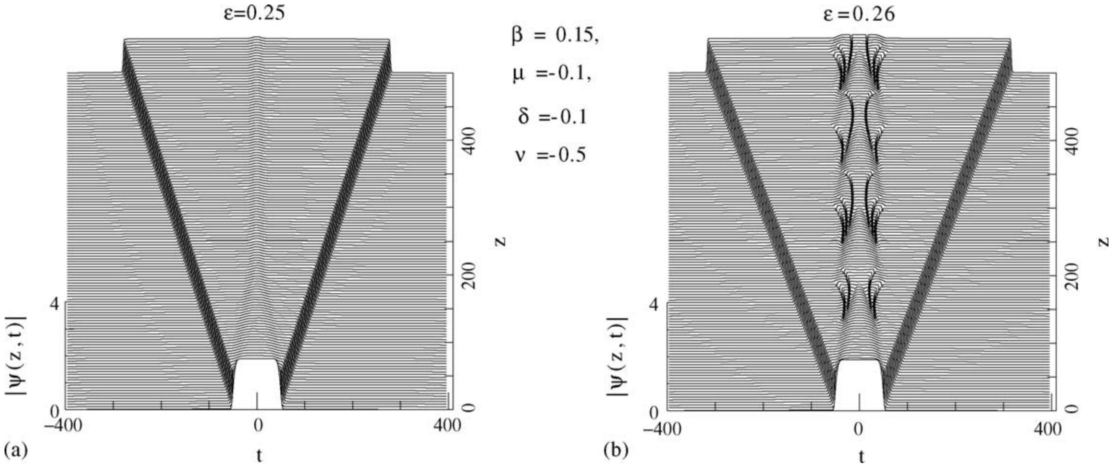

Fig. 2. Front propagation for two different values of ϵ (a) 0.25 and (b) 0.26.

fronts". These are front solutions with the profile which changes its shape chaotically but stays localized and moves with constant velocity.

Within a certain range of parameters, Eq. (1) has front solutions that are stationary and stable. We observed this in the interval $0.22 \leq \epsilon \leq 0.25$. Numerical examples are shown in Fig. 2. Due to the periodic boundary conditions of our numerical scheme, we have to deal with rectangular pulses. These are essentially two fronts propagating from or towards each other, depending on the sign of the front velocity. The continuous wave (CW) solution between the two fronts have a chirp so that the central part of the rectangular pulse must have a sink or source solution to join the two CWs.

The parameter ϵ chosen for the simulations shown in Fig. 2 admits stable fronts but not stable solitons. Solitons do exist but they are unstable in this range of ϵ values. The sink (or source) is stable as we can see from the figure but a slight change in ϵ can actually switch the solution from being stationary to being pulsating. Nevertheless fronts undergo little changes when ϵ changes from 0.25 to 0.26. Below these values and up to $\epsilon = 0.22$ fronts are stable localized structures. When ϵ increases, fronts become unstable and even chaotic.

Examples of chaotic behavior of fronts are shown in Fig. 3. Such behavior can be seen in the whole interval of values $0.3 < \epsilon < 0.4$. In analogy with exploding solitons [8] we can call these solutions as"exploding fronts". Topologically a front cannot disappear when the CW solution and the zero background are stable. Moreover, the velocity of the front is fixed and depends on the parameter ϵ. We can see from Fig. 3 that the front, indeed, moves with constant and well defined velocity. However, its shape is disturbed intermittently and attempts to recover the smooth step-like front shape are clearly visible.

As it happens in dissipative systems, there can be several stationary solutions for the same set of parameters. Fig. 3 reveals the existence of two fronts with different velocities and background CW solutions. Only one of them has the explosive instability while the other one is a stable front. This conclusion agrees with our previous observations of a multiplicity of soliton solutions for the same set of parameters [15].

Fronts and solitons can also coexist and any of them can be stable or unstable. Exact solutions for them in analytical form can be found only for a limited range of parameters [14]. In the majority of cases, exact

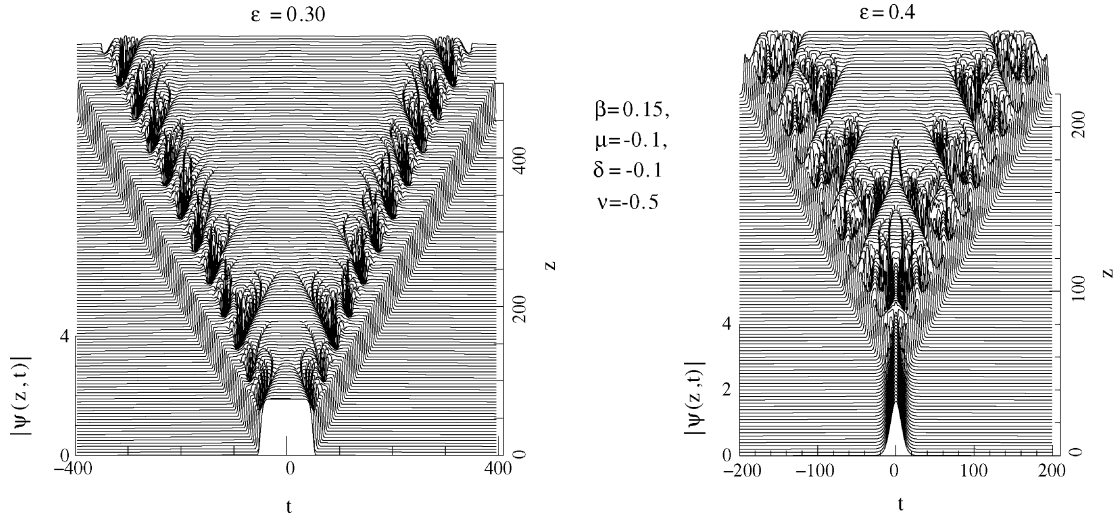

Fig. 3. Front explosions for two different values of ϵ (a) 0.3 and (b) 0.4. In (a) the initial condition is $\psi(0,t) = 1.9\ \exp(-[t/50]^{16})$ and in (b) the corresponding soliton solution for these parameters.

solutions are unknown in analytical form but they can be found numerically. If the chaotic solution is close to one of the stationary solutions for a given set of parameters, then it seems natural to study the stability of that solution. Such study for the stability of stationary soliton is presented in the following section. The stability of fronts can be studied in a similar way and will be presented elsewhere.

3. Stationary soliton solution and its stability

Let us suppose that the stationary soliton solution of the CGLE is: $\psi(z,t) = \psi_0(t)\,e^{iqz}$, where $\psi_0(t)$ is a complex function of t with exponentially decaying tails, and that q, its propagation constant, is real. This function can be easily calculated numerically. A technique for finding it has been described, for example, in Ref. [15]. The stationary front solution is a singular point of this dynamical system in an infinite-dimensional phase space. Then, the evolution of the solution in the vicinity of this singular point can be described by

$$\psi(z,t) = [\psi_0(t) + f(t)\,e^{\lambda z} + g(t)\,e^{\lambda^* z}]\,e^{iqz}, \tag{2}$$

where $f(t)$ and $g(t)$ are small perturbation functions (we assume $|f, g| \ll |\psi_0|$ for any t), and λ is the associated perturbation growth rate. In general, all λ's are complex numbers and f and g are complex functions. Substituting (2) into the CGLE (1), we obtain:

$$\begin{aligned}&(i\lambda - i\delta - q) f\,e^{\lambda z} + (i\lambda^* - i\delta - q) g\,e^{\lambda^* z} + \left(\frac{D}{2} - i\beta\right) f_{tt}\,e^{\lambda z} + \left(\frac{D}{2} - i\beta\right) g_{tt}\,e^{\lambda^* z}\\ &\quad + 3(\nu - i\mu)|\psi_0|^4 (f\,e^{\lambda z} + g\,e^{\lambda^* z}) + 2(\nu - i\mu)|\psi_0|^2 \psi_0^2 (f^*\,e^{\lambda^* z} + g^*\,e^{\lambda z})\\ &\quad + 2(1 - i\epsilon)|\psi_0|^2 (f\,e^{\lambda z} + g\,e^{\lambda^* z}) + (1 - i\epsilon)\psi_0^2 (f^*\,e^{\lambda^* z} + g^*\,e^{\lambda z}) = 0.\end{aligned} \tag{3}$$

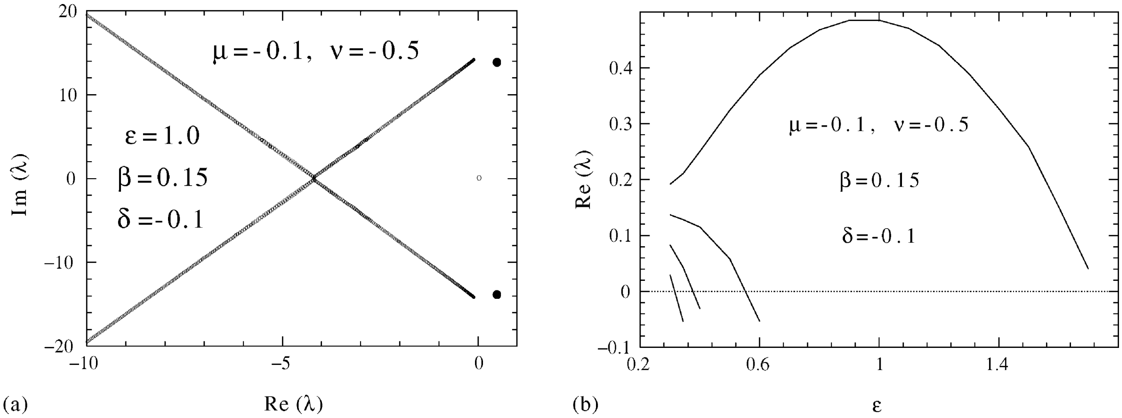

Fig. 4. (a) The spectrum of eigenvalues in the complex plane for an exploding soliton. (b) Real parts of the discrete eigenvalues (solid lines) vs. ϵ. At least one eigenvalue has a positive real part throughout the interval $0.3 < \epsilon < 1.7$. Additional eigenvalues of the discrete spectrum appear when $\epsilon < 0.6$.

Separating terms with different functional dependencies on z, we obtain the following two coupled ordinary differential equations:

$$Af + Bf_{tt} + Cg^* = \lambda f, \qquad A^*g^* + B^*g^*_{tt} + C^*f = \lambda g^*, \tag{4}$$

where

$$A = \delta - \mathrm{i}q + 2(\epsilon + \mathrm{i})|\psi_0|^2 + 3(\mu + \mathrm{i}\nu)|\psi_0|^4, \qquad B = \beta + \mathrm{i}\frac{D}{2},$$
$$C = [\epsilon + \mathrm{i} + 2(\mu + \mathrm{i}\nu)|\psi_0|^2]\psi_0^2. \tag{5}$$

The technique for solving Eq. (4) numerically has been described in Ref. [12]. This technique is essentially the same for soliton and front solutions. Here, we use the following parameters for the CGLE: $\mu = -0.1$, $\nu = -0.5$, $\beta = 0.15$ and $\delta = -0.1$, while ϵ varies from 0.3 to 1.7. In parts of this interval we observe explosions for solitons and fronts. Higher values of ϵ produce stable soliton solutions, but for lower values, at first fronts dominate and then any solution vanishes on propagation.

The complex plane, with the eigenvalues obtained as described in Ref. [12], is shown in Fig. 4a. The total spectrum consists of two complex conjugate eigenvalues with positive real part and a continuous spectrum of complex conjugate eigenvalues, all with negative real parts. We have also found that the two complex conjugate eigenvalues with positive real part turn out to be degenerate. There are two eigenfunctions corresponding to the same eigenvalue, one is an even function of t and the other is odd. The eigenvalue at the origin of the complex plane always exists, but it does not influence any dynamics.

This spectrum does not change qualitatively when we change the parameters of the system in the vicinity of the chosen point. The real part of the discrete eigenvalue is shown in Fig. 4b as a function of ϵ. The real part is much smaller than $|\mathrm{Im}(\lambda)|$ when $\epsilon > 0.6$. We can see that, when $\epsilon \approx 1$, the real part has a maximum, and no other eigenvalues appear around this point. The second eigenvalue only appears when ϵ is below 0.6. Hence, we expect that the qualitative behavior will be the same over a wide range of values of ϵ, from 0.6 to 1.7, where this eigenvalue moves to the left half of the complex plane.

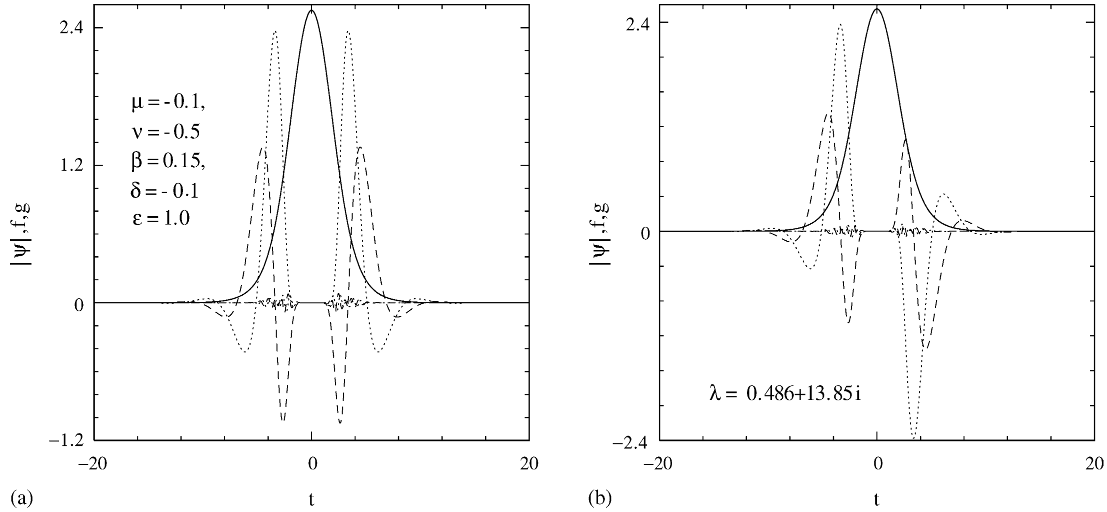

Fig. 5. Real (dotted line) and imaginary (dashed line) parts of (a) the even and (b) odd perturbation functions conveniently normalized. The solid lines in (a) and (b) show the amplitude of the soliton itself.

The whole continuous spectrum of eigenvalues is located on the left half of the complex plane. The corresponding eigenfunctions are much broader than the soliton width. These eigenfunctions are basically continuous waves, of different frequencies and wavenumbers, which are perturbed in the central zone by the soliton. In absence of the soliton, small amplitude radiation waves decay due to δ being negative. This corresponds to the pair of eigenvalues on the r.h.s. edge of the continuous spectrum with real part exactly equal to -0.1. All other eigenvalues of the continuous spectrum have real parts below -0.1 (i.e. larger than 0.1 in absolute value), due to the influence of the spectral filtering on radiation waves of different central frequencies.

As we mentioned before, each discrete eigenvalue is duplicated, or at least they coincide within the accuracy of our calculations. On the scale of Fig. 4, they are completely superimposed. The eigenfunctions corresponding to these eigenvalues are, respectively, even and odd functions of t. They are shown in Fig. 5 for the case $\epsilon = 1$. Each of these functions is nonzero mainly in the wings of the soliton and zero in the middle thus consisting distinctively of two parts. The degeneracy of the eigenvalues is lifted if the two parts become closer to each other.

4. Why do explosions happen?

In the presence of eigenvalues with positive real parts, the soliton evolution undergoes the following transformation. Suppose, initially, we have the stationary solution with small perturbations. We note that the real parts of the eigenvalues are relatively small, so that perturbations grow slowly. The imaginary parts of the eigenvalues result in oscillations simultaneously with an increase in the size of the perturbations. We also note that the soliton center is not influenced by this instability, because the eigenfunctions are almost zero in the central part of the soliton.

After the initial linear growth of the perturbation, its amplitude becomes comparable with the soliton amplitude, and the dynamics becomes strongly nonlinear. The nonlinearity mixes all perturbations, creating radiative waves. The amplitudes of radiative waves increase at the expense of the initial perturbation. Consequently, the fraction of the initial perturbation within them becomes small. The solution at this stage appears to be completely chaotic. However, the solution remains localized, both in amplitude and in width, due to the choice of the system parameters. In particular, the maximum field amplitude is limited due to the fact that μ is negative. In addition, a positive β ensures that the total width in the frequency domain also stays finite, provided that other parameters are within certain ranges. It is also important that the stationary soliton shape is fixed, thus providing the point of return.

As all radiative waves have eigenvalues with negative real parts, they decay and quickly disappear, since the eigenvalues for most of them have much larger negative real parts than the initial perturbation. This means that the solution returns to the state of a stationary soliton with a small perturbation that has an eigenvalue with positive real part. As the real part of the discrete eigenvalue is relatively small, the instability develops again later, thus repeating the whole period of the evolution described above. This process is repeated indefinitely along the z axis.

One cycle of this evolution is shown, schematically, in Fig. 6. The fixed point, shown by a black dot in this figure, corresponds to the stationary soliton solution. It can be classified as a stable–unstable focus, because all the eigenvalues in the stability analysis appear as complex conjugate pairs. We stress here that our system has an infinite number of degrees of freedom, and that the evolution actually occurs in an infinite-dimensional phase space. It cannot be reduced to a finite-dimensional problem, as all the eigenvalues play essential roles in the dynamics. As the fixed point is unstable, the trajectory leaves it in the direction in the phase space defined by the discrete eigenvalues. This motion is exponential as well as oscillatory. After complicated dynamics in the whole phase space, the trajectory, being homoclinic, returns to the same fixed point but along a different route, as defined by the continuous spectrum. This

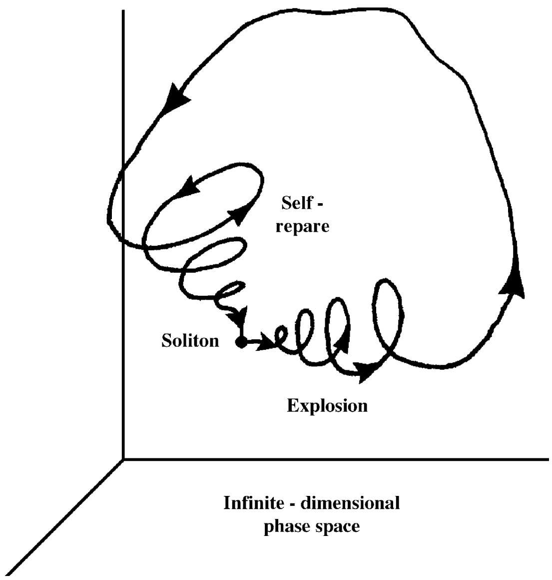

Fig. 6. One cycle of evolution of an exploding soliton in an infinite-dimensional phase space.

return is also accompanied by oscillations, as all the eigenvalues in this problem are complex. This scenario is similar to the one described by the Shil'nilov's theorem [12,16]. Similar explanations can be given for the exploding front solutions.

5. Extremely asymmetric soliton explosions

The problem of front explosions is tightly related to the problem of soliton explosions. In general, the transition from solitons to fronts happens along a certain boundary in the parameter space. Crossing this boundary when decreasing ϵ results in splitting of solitons into two fronts as shown in Fig. 3. The fronts separate and move away from the center. If explosions existed in the region of solitons they continue to exist after the transition to fronts. Let us recall that the soliton perturbation functions in Fig. 5 are nonzero at the soliton wings but close to zero in the soliton center. If the soliton splits into fronts the two parts of the perturbation function move away each attached to the corresponding front. Hence, explosions at each front can occur independently.

In the case of a soliton, the two parts of the perturbation function, namely, the left-hand side and the right-hand side perturbations are weakly interacting because of the slight overlapping between them. As a consequence, the even and odd modes of perturbation have slightly different eigenvalues. This difference was small before but appears clearly for the set of parameters given in Fig. 7. The left- and right-hand side perturbations located at the the soliton wings are still well separated. However, the difference between the eigenvalues can be well distinguished numerically. Because of this difference, the initial even symmetry of any soliton solution can be lost on propagation during the explosions. The explosions can change dramatically for the values of the system parameters where the splitting of the eigenvalues is nonzero.

Strictly speaking, explosions are never symmetric. The chaotic soliton profile during the explosions is neither even or odd function of t. It becomes close to the even function at the laminar regime of evolution. The difference between the eigenvalues of even and odd perturbation functions makes the explosions asymmetric in the sense that each explosion occurs predominantly at one side of the soliton. Even if the initial condition was symmetric and several initial explosions were at both sides of the soliton this

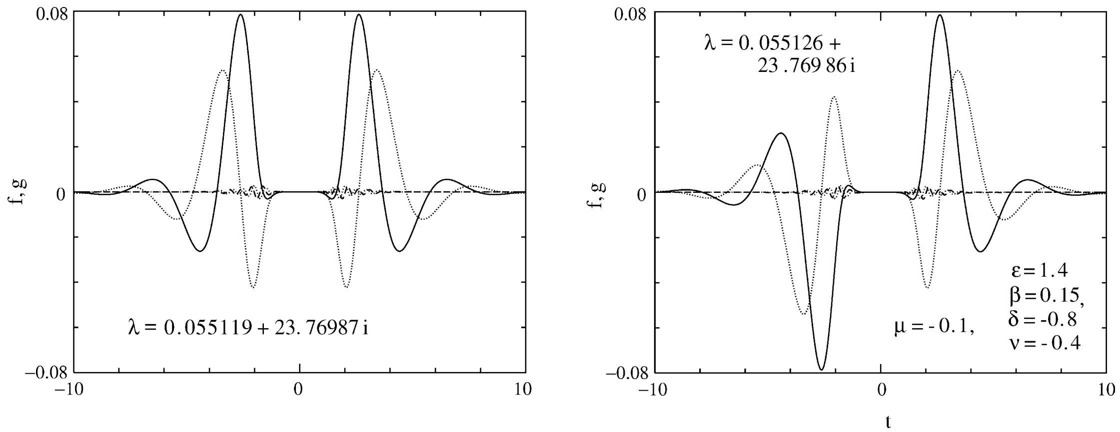

Fig. 7. (a) An even and (b) odd perturbation functions for the soliton. The eigenvalues for them are given inside each plot. The difference is of the order of 10^{-5} for the set of parameters used in this case.

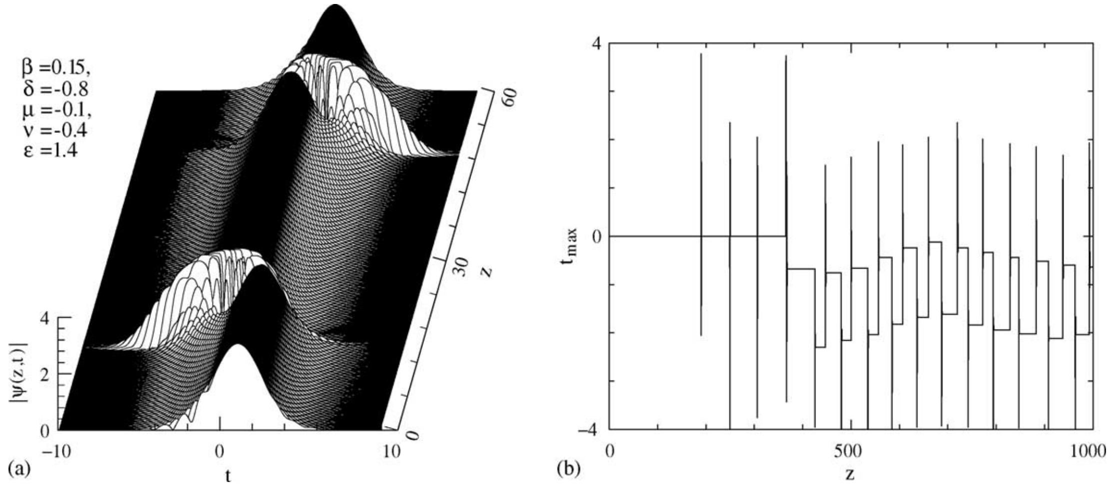

Fig. 8. (a) Extremely asymmetric soliton explosions for the set of parameters given in the figure. (b) The position of the field maxima versus z. The initial input being the unperturbed soliton solution. Extremely asymmetric explosions occur after $z = 400$.

symmetry is broken after certain propagation distance. Fig. 8 shows that for certain equation parameters a soliton can have extremely asymmetric explosions localized predominantly on one side of the pulse. We can see two of such extremely asymmetric explosions in Fig. 8a. The explosion switches the side of the soliton where it occurs after each event. This is clearly seen in Fig. 8b. The maximum of the optical field versus z shifts alternatively to the left or right-hand sides after every consecutive explosion. As a result, the position of the soliton shifts as well. The "vertical" lines in this figure indicating that at the very zenith of the explosion the peak amplitude is delocalized.

6. Conclusions

In conclusion, we have found, for the first time, exploding front solutions of the complex cubic–quintic Ginzburg–Landau (CGLE) equation. These have common features with the exploding soliton solutions. Explanations for the explosions of solitons based on the linear stability analysis of a stationary solution, are given. We have also found that, at certain values of the parameters, solitons can have extremely asymmetric explosions. These are tightly related to the exploding front solutions.

Acknowledgments

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